Fix k, $A \in SCR(k)$ $\quad X = \mathrm{Spec}(A)$ $\quad LX$.

- $HH(A) = A \otimes_k S^1 = R\Gamma(\mathrm{Map}(S^1, X), \mathcal{O})$
- $S^1 \circlearrowright HH(A) \iff HH(A)$ descends to a sheaf on BS^1

$$BS^1 \xrightarrow{f} *$$

$$\Updownarrow$$

$$pt \in \underline{\mathrm{Map}}(S^1, \underline{\mathrm{Map}}_k(HH(A), HH(A)))$$

$$= \underline{\mathrm{Map}}(S^1 \times S^1, \underline{\mathrm{Map}}_k(A, HH(A)))$$

$$= \underline{\mathrm{Map}}_k(A \otimes_k (S^1 \times S^1), A \otimes_k S^1)$$

$$(A \otimes_k (S^1 \times S^1 \longrightarrow S^1)).$$

- S^1 preserves the HKR filtration

& $\underset{\substack{\shortparallel \\ 1}}{\varepsilon} \in \underset{\rotatebox{90}{$\in$}}{\pi_1(S^1) = \mathbb{Z}}$ gives rise to

$$\varepsilon : \underset{\substack{\shortparallel \\ (\Lambda^i L_{A/k})[i]}}{gr^i_{HKR}} \longrightarrow \underset{\substack{\shortparallel \\ (\Lambda^{i+1} L_{A/k})[i]}}{gr^{i+1}_{HKR}[-1]}$$

is (the shift of) d.

- $HC^- = (HH(A))^{hS^1} = Rf_*(HH(A))$

Goal: extend all of these from $*$ to $[\mathbb{A}^1/\mathbb{G}_m]_k$

Bhargav: $\underset{\substack{\| \\ \mathrm{Spec}(HH(A/k))}}{LX}$ extends to $L_{Fil}X = \mathrm{Map}(S^1_{Fil}, X)$

$$\begin{tikzcd}
\mathrm{Map}(S^1_{Fil}, X) \arrow[d] \\
\mathrm{Map}(S^1_{Fil}\times S^1_{Fil}, X) \arrow[d, phantom, "\|"] \\
\mathrm{Map}(S^1_{Fil}, L_{Fil}X)
\end{tikzcd}$$

- there is an S^1_{Fil} action on $L_{Fil}X$

$$\begin{tikzcd}
& S^1_{Fil}\times_{[\mathbb{A}^1/\mathbb{G}_m]} L_{Fil}X \arrow[rr, "\circ"] \arrow[dl] \arrow[dr] & & L_{Fil}X \arrow[dl] \\
S^1_{Fil}\times \mathrm{Map}(S^1_{Fil}, L_{Fil}X) \arrow[urrr, bend left] & & {[\mathbb{A}^1/\mathbb{G}_m]} & & 1 = * \arrow[ll]
\end{tikzcd}$$

$$\begin{tikzcd} L_{Fil}X \arrow[r] \arrow[d, "\beta"] & L_{Fil}X/S^1_{Fil} \arrow[d, "\alpha"] \\ {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]/S^1_{Fil}} \end{tikzcd}$$

Prop. α is an affine map of global coh. dim ≤ 0

pf sketch: affine ✓ b/c β is affine

$$QCoh(K(H;2)) = \lim_{[n]\in\Delta} QCoh((H)^{n^2}).$$

since all the transition map $H^{n^2} \longrightarrow H^{m^2}$ are flat
$(m \geq n)$

$$\text{coh deg} \leq 0 = \lim(\text{coh deg} \leq 0).$$

so it suffices to show $\mathcal{O}(L_{Fil}X)$ is in ≤ 0
which is verified in Bhargav's talk.

Fact: this implies the map α satisfies Beck–Chevalley property:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & L_{Fil}X / S^1_{Fil} \arrow[d, "\alpha"'] \\ Y \arrow[r] & [\mathbb{A}^1/\mathbb{G}_m] / S^1_{Fil} \end{tikzcd}$$

Cor: $\alpha_* \mathcal{O}$ is the $\underbrace{(HH_{A/k}, HKR, S^1\text{-action})}_{\text{triple}}$

$$
\begin{tikzcd}
 & (\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{gr}} \arrow[rr, "\tilde{0}"] \arrow[dl, "\rho^{\mathrm{gr}}"'] \arrow[dd, "p^{\mathrm{gr}}"] & & \mathsf{L}_{\mathsf{Fil}}X \arrow[dl, "\rho"] \arrow[dd, "p"] & & (\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{u}} \arrow[ll, "\tilde{1}"'] \arrow[dl, "\rho^{\mathrm{u}}"] \arrow[dd, "p"] \\
(\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{gr}}/(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{gr}} \arrow[rr, "\tilde{0}_a"] \arrow[dd, "a^{\mathrm{gr}}"] & & \mathsf{L}_{\mathsf{Fil}}X/\mathsf{S}^1_{\mathsf{Fil}} \arrow[dd, "a"] & & (\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{u}}/(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{u}} \arrow[ll, "\tilde{1}_a"'] \arrow[dd, "a^{\mathrm{u}}"] & \\
 & B\mathbb{G}_m \arrow[rr, "0"] \arrow[dl, "e^{\mathrm{gr}}"] & & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[dl, "e", "(j)"'] & & * \arrow[ll, "1"'] \arrow[dl, "e^{\mathrm{u}}"] \\
B(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{gr}} \arrow[rr, "0_a"] \arrow[d, "\pi^{\mathrm{gr}}"] & & B\mathsf{S}^1_{\mathsf{Fil}} \arrow[d, "\pi"] & & B(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{u}} \arrow[ll, "1_a"'] \arrow[d, "\pi^{\mathrm{u}}"] & = \mathrm{Aff}(BS^1) \\
B\mathbb{G}_m \arrow[rr, "0"] & & {[\mathbb{A}^1/\mathbb{G}_m]} & & * \arrow[ll, "1"'] &
\end{tikzcd}
$$

$$
R\mathrm{Hom}_{\Lambda}(k, DR) \qquad \cdots \to \Lambda\{1\} \to \Lambda \to k
$$

$$
= DR \to DR\{1\} \to DR\{2\} \to \cdots
$$

Defn: $HC^-_{Fil}(X) := \pi_* \alpha_* \mathcal{O}$

Immediate: $\exists$ natural map

$$(HC^-_{Fil}(X))^u \longrightarrow HC^-_k(A)$$

E_∞-alg in $QCoh(B(S^1_{Fil}))^{gr}$

$\parallel$

Construction: $DR(A/k) = \mathrm{Sym}^*_A(\mathbb{L}_{A/k}[1]) \in CAlg(Mod_\Lambda^\otimes)$

wt 1

with differential (of deg -1)

$$A \xrightarrow{d} \mathbb{L}_{A/k} \xrightarrow{d} \Lambda^2 \mathbb{L}_{A/k} \longrightarrow \cdots$$

Recall in Arpon's talk:

$$QCoh(B(S^1_{Fil})^{gr})^{\otimes} = (D(Mod^{str}_{\Lambda}))^{\otimes}.$$

$$\Lambda := k \oplus k \cdot \varepsilon[1]$$

deg -1 or wt 1

Prop: $(\alpha_* \mathcal{O})^{gr} = DR(A/k)$

pf sketch: Bhargav explained why the underlying graded cplxes are the same

need to check that the differential matches

- reduce to $A = k[x_1, \dots, x_n]$.
- $k = \mathbb{Z}_{(p)}$
- everything in sight lives in coh. deg 0 & p-torsion free, $k = \mathbb{Q}_p$

Prop. when A satisfies: $gr^{i}(\alpha_*(\mathcal{O}))$ is uniformly left bdd,

then $HC^{-}_{fil}(A)^{u} \xrightarrow{\cong} HC^{-}(A)$

holds if A is smooth - k - alg.

Prop. wt i of $\left(HC^-_{\mathrm{Fil}}(A)\right)^{gr}$ is

$$\prod_{q \geq i} \left(\Lambda^q \mathbb{L}_{A/k}\right)[2i-q].$$

with differential $= d_{A/k} + d_{dR}$.

$$\mathrm{Fil}^i_H \widehat{dR}_{A/k}[2i]$$