

Fix  $k$ ,  $A \in \text{SCR}(k)$   $X = \text{Spec}(A)$   $\swarrow$  LX.

•  $\text{HH}(A) = A \otimes_k S' = R\Gamma(\text{Map}(S', X), \mathcal{O})$

•  $S' \hookrightarrow \text{HH}(A) \iff \text{HH}(A)$  descends to a sheaf

on  $B S'$

$$\text{pt} \in \underline{\text{Map}}(S', \underline{\text{Map}}_k(\text{HH}(A), \text{HH}(A)))$$

$$= \underline{\text{Map}}(S' \times S', \underline{\text{Map}}_k(A, \text{HH}(A)))$$

$$= \underline{\text{Map}}_k(A \otimes_k (S' \times S'), A \otimes_k S')$$

$(A \otimes_k (S' \times S' \rightarrow S'))$

•  $S'$  preserves the HKR filtration

&  $\varepsilon \in \pi_1(S') = \mathbb{Z}$  gives rise to

$$\varepsilon: \text{gr}_{\text{HKR}}^i \longrightarrow \text{gr}_{\text{HKR}}^{i+1}[-1]$$

$$\left( \bigwedge_{A/k}^i L \right) [i] \longrightarrow \left( \bigwedge_{A/k}^{i+1} L \right) [i]$$

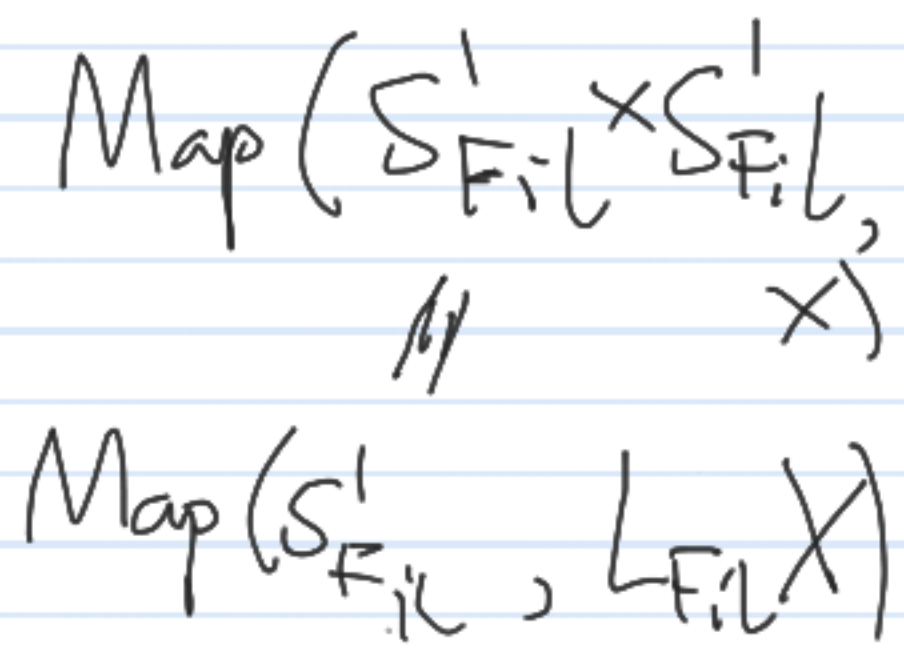
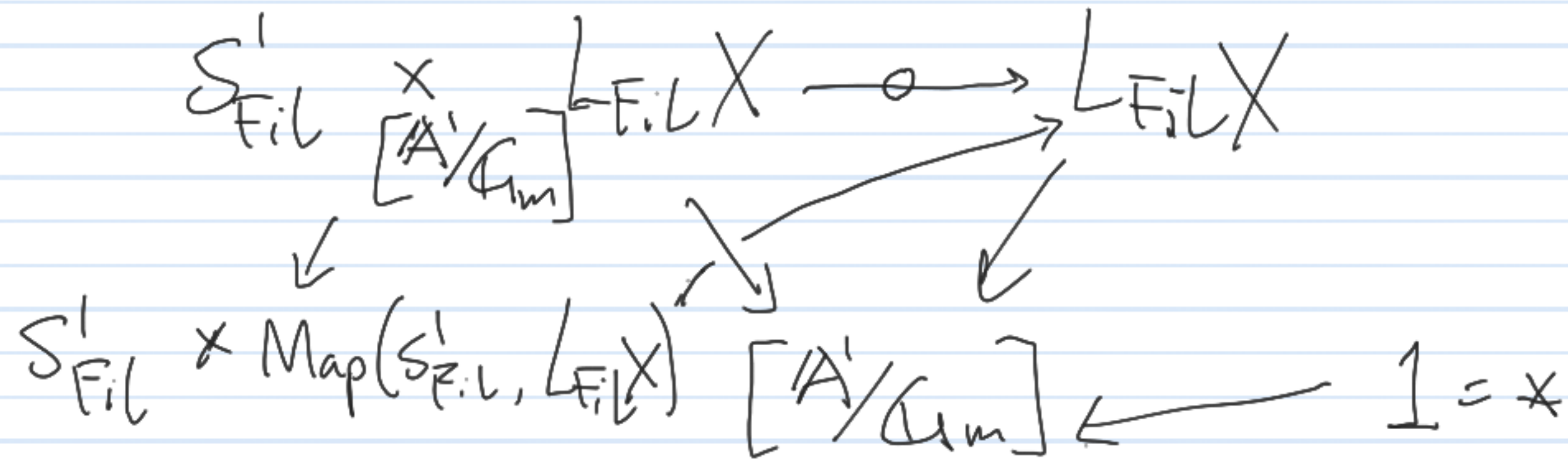
is (the shift of)  $d$ .

•  $H\mathcal{C}^- = (HH(A))^{hS'} = Rf_* (HH(A))$

Goal: extend all of these from  $*$  to  $[A'/G_m]_k$

Bhargava:  $LX \underset{\text{Spec}(HH(A)_k)}{\parallel} L_{\text{Fil}}X = \text{Map}(S'_{\text{Fil}}, X)$

• there is an  $S'_{\text{Fil}}$  action on  $L_{\text{Fil}}X$ .



$$\begin{array}{ccc} L_{\text{Fil}X} & \longrightarrow & L_{\text{Fil}X}/S'_{\text{Fil}} \\ \downarrow \beta & & \downarrow \alpha \\ [A'_{\text{Gim}}] & \longrightarrow & [A'_{\text{Gim}}]/S'_{\text{Fil}} \end{array}$$

Prop.  $\alpha$  is an affine map of global coh.  $\dim \leq 0$ .

of sketch: affine  $\vee$  b/c  $\beta$  is affine

$$\text{QCoh}(K(H; Z)) = \lim_{[n] \in \Delta} \text{QCoh}((H)^{n^2})$$

since all the transition map  $H^{n^2} \rightarrow H^{m^2}$  are flat

$$\text{coh deg} \leq 0 = \lim_{(m \geq n)} (\text{coh deg} \leq 0)$$

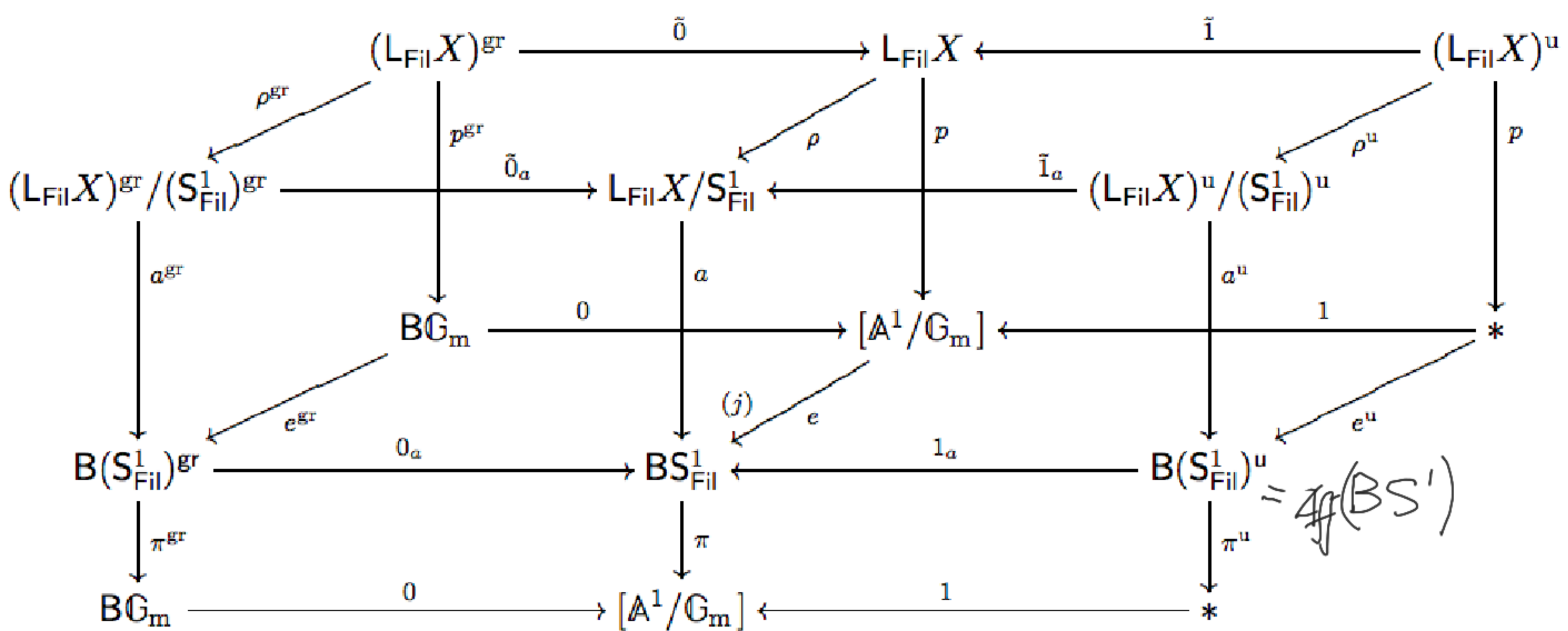
so it suffices to show  $\mathcal{O}(L_{\text{Fil}} X)$  is in  $\leq 0$ .  
which is verified in Bhargava's talk.

Fact: this implies the map  $\alpha$  satisfies

Beck-Chevalley property:

$$\begin{array}{ccc} Z & \longrightarrow & L_{F: L/X} / S'_{F: L} \\ \downarrow & & \downarrow \alpha \\ Y & \longrightarrow & [A'/G_m] / S'_{F: L} \end{array}$$

Cor:  $\alpha_* \mathcal{O}$  is the  $(\text{HH}_{A/k}, \text{HKR}, S'\text{-action})$   
triple



$R\text{Hom}_{\Delta}(k, \text{DR}) \rightarrow \Delta\{1\} \rightarrow \Delta \rightarrow k$   
 $= \text{DR} \rightarrow \text{DR}\{1\} \rightarrow \text{DR}\{2\} \rightarrow \dots$

Defn:  $HC_{\text{Fil}}^-(X) := \pi_* \alpha_* \mathcal{O}^1$

Immediate:  $\exists$  natural map

$$(HC_{\text{Fil}}^-(X))^u \longrightarrow HC_k^-(A) \quad \begin{array}{l} \text{E}_\infty\text{-alg in} \\ \text{QCoh}(B(S_{\text{Fil}}^1)^{\text{gr}}) \\ \parallel \end{array}$$

Construction:  $DR(A/k) = \text{Sym}_A^*(L_{A/k}[1]) \in \text{CAlg}(\text{Mod}_A)$

with differential  
(of deg -1)

$$A \xrightarrow{d} L_{A/k} \xrightarrow{d} \wedge^2 L_{A/k} \longrightarrow \dots$$

← wt 1



Recall in Arpon's talk:

$$\mathbb{Q} \text{ Coh}(B(S_{\text{Fil}}^1)^{\text{gr}})^{\otimes} = \left( \text{Mod}_{\Lambda}^{\text{str}} \right)^{\otimes}$$

$$\Lambda = k \oplus k \cdot \mathbb{E}[1]$$

deg -1 or wt 1

$$\text{Prop: } (\alpha_* \mathcal{O})^{\text{gr}} = \text{DR}(A/k)$$

pf sketch: Bhargava explained why the underlying graded cplxes are the same

need to check that the differential matches

- reduce to  $A = k[x_1, \dots, x_n]$

- $k = \mathbb{Z}(\varphi)$

- everything in sight lives in coh. deg 0 &

$p$ -torsion free,  $k = \mathbb{Q}_p$ .

Prop. when  $A$  satisfies:  $gr^i(\alpha_* \mathcal{O})$  is <sup>uniformly</sup> left bdd,

then  $HC_{fil}^-(A)^u \xrightarrow{\cong} HC^-(A)$

holds if  $A$  is smooth- $k$ -alg.

Prop. wt  $\bar{i}$  of  $(\mathrm{HC}_{\mathrm{Fil}}^-(A))^{\mathrm{gr}}$  is

$$\prod_{g \geq \bar{i}} \left( \bigoplus_{A/k}^g [2i - g] \right)$$

with differential  $= d_{A/k} + d_{dR}$ .

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$$\mathrm{Fil}_{\mathrm{H}}^{\bar{i}} \widehat{dR}_{A/k} [2i]$$